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Regular affine tilings and regular maps on a flat torus

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Abstract

A regular affine tiling of a flat (locally isometric to a euclidean plane) torus is defined to be the affine image of a tiling of a flat torus with congruent regular p-gons, adjacent ones sharing a side. Only triangular, hexagonal, and quadrangular affine tilings exist. Each tiling is determined up to a shift by its set of rotation numbers and its multiplicity. Criteria are given for two tilings to be affine images of each other. The usual codes are calculated from the rotation numbers and multiplicity. The results are extended to regular toroidal maps.

1. Introduction

Tilings [8] are a topic of general interest and tilings of a *torus* with *hexagons* or with *triangles* have been studied extensively in both mathematics [1,2,16,20,25], and physics/chemistry [3,4,9,11–15,22,23]. Toroidal hexagonal tilings, also called toroidal *polyhexes*, have recently become of practical importance with the likely discovery of toroidal carbon cages [17]. The tilings studied in this paper are also called *tessellations* or *patterns* or *maps* or *convex cell complexes*. Concerning the general theories of tilings and of maps we refer the reader to [5,7,8,10]. Higher-dimensional polytopes are studied, for example, in [6,18], and [19].

A *regular euclidean tiling of a flat torus* is, by definition, a tiling of a *flat* (locally isometric to a euclidean plane) torus with congruent regular euclidean p-gons in such a way that adjacent p-gons share a side. Simple geometric arguments show that p must be 3,4, or 6, in which cases each vertex is contained in 6, 4, or 3 edge ends, respectively. We define a *regular affine tiling of a flat torus* to be the image of a regular euclidean tiling under an affine isomorphism of flat tori. Such an *affine isomorphism* is a one-to-one mapping between flat tori that maps geodesics to geodesics. The *dual* of a regular affine toroidal tiling is obtained by connecting the centroids (centers of

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mass) of adjacent tiles with a geodesic across a common edge. The dual of a p-gonal affine toroidal tiling is a p'-gonal affine toroidal tiling, where $p' = 6, 4$, or 3 according as $p = 3, 4$, or 6. The second dual coincides with the original tiling.

We consider all tilings on one and the same flat torus $\mathscr{T}$. The edges of a TA (QA) (= triangular (quadrangular) regular affine tiling of $\mathscr{T}$) form 3 (2) families of equidistant parallel closed geodesics. The *multiplicity* of a TA (QA) is defined to be the gcd (= greatest common divisor) of the numbers of geodesics in each of the three families (the assignment, to each of the two families, of the number of geodesics in the family). We show that two TAs (QAs) can be obtained from each other by a shift if and only if every geodesic cycle of one tiling is parallel to such a cycle of the other tiling and the multiplicities of the tilings coincide (in the case of a QA this means that the numbers of geodesics in parallel families are the same). This is quite obvious for QAs but requires a proof in the case of TAs. It also extends to HAs (= hexagonal regular affine tilings of $\mathscr{T}$) by duality. Any three (two) distinct directions combined with any positive integer (any asignment of positive integers to the directions) occur as the set of directions of geodesic cycles, combined with the multiplicity, of a TA (QA). An obvious consequence of the results is that if the *directions* of the geodesic cycles in a TA are *given*, then the numbers of cycles in each of the three families of equidistant geodesic cycles cannot be chosen independently: their gcd determines them.

To coordinatize regular affine tilings of $\mathscr{T}$, we single out two *oriented* closed geodesics on $\mathscr{T}$, say Ξ and Υ, with a single point of intersection and denote by $\xi, \upsilon, 0 \leqslant \xi, \upsilon < 1$, the coordinates of a point on the torus with respect to the axes Ξ, Υ. The slope $\Delta\upsilon/\Delta\xi$ of a closed geodesic γ of $\mathscr{T}$, which is a rational number when finite, is called the *rotation number* of γ with respect to Ξ, Υ (= the ratio of the numbers of turns in the Υ and Ξ directions in one traversal of γ). Then the rotation numbers of a regular affine tiling of $\mathscr{T}$ are defined. In coordinate form, the results in the preceding paragraph state that two regular affine tilings on $\mathscr{T}$ are shifts of each other if and only if their rotation numbers and multiplicities are the same. It follows immediately that two affine tilings of $\mathscr{T}$ are affine isomorphic (= they are affine isomorphic images of each other) if and only if their rotation numbers can be transformed into each other by a unimodular integer 2×2 matrix and they have the same multiplicity (which is compatible with the transformation of rotation numbers in the case of a QA). (The action of a 2×2 matrix on the ratio of two integers is the fraction determined by the product of the matrix and the column vector of the fraction.)

The above results extend to regular maps on $\mathscr{T}$. Such maps are homeomorphic images of regular affine tilings of $\mathscr{T}$.

We treat tilings differently from [1,2,16,12,13,20] in that we fix a torus and vary the tilings as opposed to fixing a tiling of the euclidean plane and then defining tori by various identifications. An accompanying feature is the unique coordinatization of the shift-isomorphism (isotopy) classes of regular affine toroidal tilings (regular toroidal maps). In the papers cited a set of 6 (4) possibly coinciding codes is assigned to each affine isomorphism class of a TA (QA). We determine these codes from the rotation numbers and the multiplicity. Isotopy classes of hexagonal toroidal tilings have also been studied in [14] in connection with chemical isomerism.

The next section contains general preliminaries about tori. In the following sections we treat QAs, TAs, and HAs, in that order, and include Section 5 on a special class of TAs. In the last section the results are extended to regular maps of a torus.

2. Preliminaries

This section contains standard material on tori and on their closed geodesics, which is used in the subsequent sections.

Let $\mathscr{T}=[0,1)\times[0,1)=\{(\xi,\upsilon)\colon 0\leqslant\xi,\upsilon<1\}, pr(x,y)=pr(x+i,y+j)$, $x,y\in\mathbb{R}, i,j\in\mathbb{Z}, pr|_{\mathscr{T}}=id$. The mapping pr defines the quotient topology and the metric of a flat torus on $\mathscr{T}$ from the topology and the metric of the (x,y)-plane. The geodesics of $\mathscr{T}$ are the images of straight lines under pr. Let us orient the geodesics $\Xi=[0,1)\times\{0\}$, $\Upsilon=\{0\}\times[0,1)$ by the increasing order in $[0,1)$. These geodesics have a single point of intersection. It is easy to see that if $\mathscr{T}'$ is another flat torus and Ξ', Υ' are oriented closed geodesics on $\mathscr{T}'$ with a single point of intersection, then there is an affine isomorphism of $\mathscr{T}$ onto $\mathscr{T}'$ that transforms Ξ into Ξ' and Υ into Υ', and preserves orientations. Therefore, there is no loss of generality in considering regular affine tilings only on $\mathscr{T}$, equipped with Ξ, Υ. Unless stated otherwise, every tiling in this paper is on $\mathscr{T}$. However, we note that $\mathscr{T}$ and $\mathscr{T}'$ as above may not be isometrically mapped onto each other and the angle between Ξ' and Υ' may not be $\pi/2$.

Let γ be a closed geodesic of $\mathscr{T}$. The straight lines σ of the (x,y)-plane for which $\gamma=pr(\sigma)$, called the *covering lines* of γ, form a family of equidistant parallel lines. The *rotation number* of γ is defined to be the slope $\Delta y/\Delta x$ of any of these lines. It is an element of $\mathbb{Q}\cup\{\infty\}$, where $\mathbb{Q}$ is the set of rational numbers. Let us traverse γ in one of the two directions once. Then the rotation number of γ is the number of turns in the υ-direction (meridian direction) divided by the number of turns in the ξ-direction (parallel direction). Since γ is a simple curve, these two numbers of turns are relatively prime to each other. We obtain their negatives when γ is traversed in the opposite direction. Note that γ itself is a family of equidistant parallel line segments of $[0,1)\times[0,1)$.

It is easy to see that the affine automorphisms of $\mathscr{T}$ are exactly the quotients, by pr, of planar transformations of the form $(x,y)\to a_{(x_0,y_0)}a_U(x,y)$ with $a_{(x_0,y_0)}(x,y)=(x+x_0,y+y_0)$ and $a_U(x,y)=(\bar{x},\bar{y})$, where $(\bar{y},\bar{x})=(y,x)U^t$ and $(y,x)U^t$ is the product of the row vector (y,x) with the transpose U^t of a unimodular $(=\det U=\pm 1)$ integer 2×2 matrix U. The dependence of a_U on U is defined in this roundabout manner because then the slope of the image $a_U\sigma$ of a line σ with slope $s=m/p$ is Us, where Us here and later denotes the fraction determined by the column vector $U\langle m,p\rangle^t$. Clearly, Us is in reduced form whenever s is. We shall also use the notation $\alpha_{pr(x_0,y_0)}=pra_{(x_0,y_0)}, \alpha_U=pra_U$. Then $\alpha_{pr(x_0,y_0)}$ is a *shift*.

Since a unimodular *integer* matrix appears in the definition of an affine automorphism of $\mathscr{T}$, the group $\mathfrak{A}$ of these affine automorphisms splits into as many connected components as there are such matrices. The component $\mathfrak{A}_0$ containing the identity transformation *id* is the group of shifts of $\mathscr{T}$. Since $\mathfrak{A}_0$ consists of isometries, it is the group of rigid motions of $\mathscr{T}$.

The first of the two lemmas below is well known [24, pp. 208–209] and so is probably the second, as well. The first lemma reveals the structure of the intersection of two nonparallel closed geodesics of $\mathscr{T}$. In the proof we use the observation that any piece of length $\sqrt{m^2+p^2}$ of a covering line of a closed geodesic with rotation number $m/p, \gcd(m,p)=1$, covers the geodesic exactly once. The second lemma is of strictly algebraic nature.

Lemma 1. *Let m/p and M/P be two distinct elements of $\mathbb{Q}\cup\{\infty\}$ in reduced form and let $\lambda = pM - Pm$. Any two closed geodesics of $\mathscr{T}$ with rotation numbers m/p and M/P intersect each other at $|\lambda|$ points.*

Proof. It is easy to see that the closed geodesics divide the torus into congruent parallelograms in parallel position. Let v denote the number of these parallelograms, which is also the number of points of intersection of the closed geodesics. The plane vectors $\langle p/v, m/v\rangle, \langle P/v, M/v\rangle$ determine a parallelogram that is congruent to a parallelogram in the partition. The area of this parallelogram is $|pM-Pm|/v^2 = |\lambda|/v^2$. Since the total area is equal to 1, $v|\lambda|/v^2 = |\lambda|/v = 1$. Therefore, $v=|\lambda|$. □

Lemma 2. *Let*

$$M = \begin{bmatrix} m_1 & m_2 \\ p_1 & p_2 \end{bmatrix},$$

$\lambda = \det M \neq 0$, *with* $m_1, p_1, m_2, p_2 \in \mathbb{Z}, \gcd(m_I, p_I) = 1, I = 1,2$. *For each of* $+$ *and* $-$ *there exists a unique integer matrix*

$$N^{\pm} = \begin{bmatrix} 1 & r^{\pm} \\ 0 & s^{\pm} \end{bmatrix}$$

such that $N^{\pm}M^{-1}$ *is unimodular integer and* $0 \leqslant r^+ < s^+$, $0 \geqslant r^- > s^-$. *Then* $s^{\pm} = \pm|\lambda|$, $r^+ \equiv r^- (\operatorname{mod} |\lambda|), \operatorname{sign}\det(N^{\pm}M^{-1}) = \pm\operatorname{sign}\lambda$. *If M is replaced with $-M$, then $N^{\pm}$ do not change. If a column of M is multiplied by -1, then $-r^{\pm}, -s^{\pm}$ become $r^{\mp}, s^{\mp}$.*

Proof. Necessarily, $s^{\pm} = \pm|\lambda|$ and $r^{\pm}$ is the unique integer of the form $r^{\pm} = em_2 + fp_2, em_1 + fp_1 = 1$, such that $0 \leqslant r^+ < |\lambda|$ $(-|\lambda| < r^- \leqslant 0)$. ($N^+$ is a *Hermite normal form* [21].) All statements of the lemma are obvious or follow from this. □

In the last section we deal with the topological analogues of the above notions and results (compare [24]). The topological analogue of a closed geodesic (an affine automorphism, a shift) of $\mathscr{T}$ is a *simple* closed curve (a self-homeomorphism, a self-homeomorphism isotopic to the identity) of $\mathscr{T}$. Homeomorphisms h_0, h_1 from a topological space $\mathscr{S}$ *into* a topological space $\mathscr{S}'$ are said to be *isotopic* to each other if they can be continuously deformed into each other through homeomorphisms, that is, if a continuous mapping $h(x,\tau): \mathscr{S} \times [0,1] \to \mathscr{S}'$ exists such that $h(\cdot,0) = h_0, h(\cdot,1) = h_1$ and each $h(\cdot,\tau)$ is a homeomorphism of $\mathscr{S}$ *into* $\mathscr{S}'$. The isotopy classes of self-homeomorphisms of $\mathscr{T}$ form a group every element of which contains affine automorphisms differing from each other only by shifts. *Simple closed curves* γ_1, γ_2 in a topological space are regarded as *isotopic* if there are isotopic homeomorphisms of a circle onto them. If two *oriented* simple closed torus curves Ξ_t, Υ_t that meet and cross at a single point, are chosen, then the isotopy class of an *unoriented* simple closed torus curve γ can be described by its rotation number $s = m/p, \gcd(m,p) = 1$ ($s = \infty = 1/0 = -1/0$ allowed), where m (p) is the number of turns in the Υ_t-direction (Ξ_t-direction) as γ is traversed once in a given direction. (In the other direction we obtain $-m$ and $-p$.)

3. Shifts and affine isomorphisms of QAs

In this section we cooordinatize the shift-isomorphism classes of QAs in a unique manner. The affine isomorphism classes of QAs are given in several works; see, for example, [1,2,11–13,20,16]. In these papers 4 possibly coinciding codes are assigned to every affine isomorphism class of QAs; affine non-isomorphic QAs do not share a code. We use the codes of [16]. We calculate them from the shift-invariant ($=$ geometric) code of a QA.

An intrinsic definition of a QA is this: It consists of parallelograms congruent to a given parallelogram. The tiles are in parallel position and are adjoined along common sides.

Let Q be a QA and let $\mathfrak{C}_1, \mathfrak{C}_2$ denote the two families of parallel geodesic cycles of Q. We denote by q_i the rotation number of a geodesic cycle in $\mathfrak{C}_i$ and by μ_i the number of geodesic cycles in $\mathfrak{C}_i$. To Q we assign the pair (q,μ), where q is the set $q = \{q_1, q_2\}$ and $\mu: q \to \mathbb{N}$ is the function for which $\mu(q_i) = \mu_i$, $i = 1,2$. The q_i are called the *rotation numbers* of Q, the function μ the *multiplicity* of Q, and (q,μ) the *geometric code* of Q. The first two statements in the theorem below are quite obvious. The last statement follows from Lemma 1. We leave the details to the reader.

Theorem 3. *Two* QA*s have the same geometric code if and only if they are shifts of each other. Every pair* (q,μ), *where* q *is a two-element subset of* $\mathbb{Q} \cup \{\infty\}$ *and* μ *is a function* $q \to \mathbb{N}$, *is the geometric code of a* QA. *The number of vertices of a* QA *with code* (q,μ), $q_i = m_i/p_i$, $\gcd(m_i, p_i) = 1$, $\mu: q_i \to \mu_i$, $i = 1,2, \lambda = m_1 p_2 - m_2 p_1$, *is* $n = |\lambda|\mu_1\mu_2$, *and every* q_j*-cycle contains* $|\lambda|\mu_i$ *vertices*, $i \neq j$.

In the rest of the paper we denote by $Q(q,\mu)$ any QA, or the class of QAs, with geometric code (q,μ).

The image αQ of a QA Q under an affine automorphism α of $\mathscr{T}$ is obviously a QA, as well. We say that QAs Q, Q' are affine isomorphic if there is an affine automorphism α of $\mathscr{T}$, such that $Q' = \alpha Q$.

If α is an affine automorphism of $\mathscr{T}$, associated with U, then $\alpha Q(q, \mu) \in Q(Uq, U\mu)$, where $Uq = \{Us\colon s \in q\}$ and $U\mu$ is the function $Us \to \mu(s), s \in q$ (see Section 2 for a definition of Us). Conversely, by Theorem 3, any element of $Q(Uq, U\mu)$ is of the form $\alpha Q(q, \mu)$, where α is an affine automorphism of $\mathscr{T}$, associated with U. Therefore, $\bigcup_U Q(Uq, U\mu)$ is the affine isomorphism class of $Q(q, \mu)$.

The definition of the *affine* ("affine" is our term) codes of a QA Q in [16] is equivalent to the following: Let us choose an ordering $\mathfrak{C}_i, \mathfrak{C}_j$ of $\mathfrak{C}_1, \mathfrak{C}_2$ and an orientation o_I on $\mathfrak{C}_I$, $I = 1,2$. In the code (l_i, c_{ij}, μ_i) belonging to these choices, (1) l_i is the length, in the graph of Q, of an element of $\mathfrak{C}_i$ and (2) c_{ij} is the length, in the graph of Q, of the path along a geodesic $\Gamma_i \in \mathfrak{C}_i$ from a vertex $v \in \Gamma_i$ in the direction o_i to the first point of intersection, with Γ_i, of the geodesic $\Gamma_j \in \mathfrak{C}_j$ through v, moving from v on Γ_j in the direction o_j. The pairs of orientations o_1, o_2 and $-o_1, -o_2$ give the same c_{ij} and $c_{ij} + c'_{ij} = 0 \,(\mathrm{mod}\, l_i)$ for the c'_{ij} obtained for $o_1, -o_2$ in place of o_1, o_2.

The code (l_i, c_{ij}, μ_i) can be obtained from $(q_1, q_2, q_i \to \mu_i)$ in this way: Let $q_I = m_I/p_I$, $\gcd(m_I, p_I) = 1$, $I = 1,2$, $\lambda = m_1 p_2 - m_2 p_1$, and let

$$M_{ij} = \begin{bmatrix} m_i & m_j \\ p_i & p_j \end{bmatrix}.$$

By Lemma 2, there is a unique

$$N_{ij} = \begin{bmatrix} 1 & r_{ij} \\ 0 & s_{ij} \end{bmatrix},$$

$0 \leqslant r_{ij} < s_{ij}$, such that $N_{ij} M_{ij}^{-1}$ is unimodular integer. In general, for r_{ij} we obtain two values r_{ij}, r'_{ij} satisfying $r_{ij} + r'_{ij} = 0 \,(\mathrm{mod}\, |\lambda|)$ because no choice of the signs of $m_I, p_I, I = 1,2$, has been specified. The affine automorphism $\alpha_{N_{ij} M_{ij}^{-1}}$ maps the cycles of Q with orientation determined by the plane vector $\langle p_i, m_i \rangle$ ($\langle p_j, m_j \rangle$) to cycles of the tiling $N_{ij} M_{ij}^{-1} Q$ with orientation determined by the plane vector $\langle 0, 1 \rangle$ ($\langle s_{ij}, r_{ij} \rangle$). By Theorem 3, $l_i = |\lambda| \mu_j$, and thus $\{c_{ij}, c'_{ij}\} = \{r_{ij} \mu_j, r'_{ij} \mu_j\}$.

4. Shifts and affine isomorphisms of TAs

In this section we coordinatize the shift-isomorphism classes of TAs in a unique manner. We find that if the slopes of geodesic cycles are given, then the numbers of cycles in each of the three families of parallel geodesic cycles is determined by their gcd. At the end of the section, affine isomorphisms of TAs are considered briefly analogously to QAs.

A TA T can alternately be defined in this way: T is a tiling of $\mathscr{T}$ with congruent triangles in such a way that adjacent triangles share a common side and are rotated a half-turn with respect to each other. This definition is equivalent to the one in the

Introduction because a new metric can be introduced on $\mathscr{T}$ so that T becomes a regular euclidean tiling.

Another way to define a TA is this: we consider a QA on $\mathscr{T}$ and draw one of the two diagonals of a tile and all parallel diagonals. This gives a TA and a third family of equidistant parallel geodesic cycles.

By the preceding paragraph, every TA T decomposes into a family of parallel geodesic cycles in three ways. The *multiplicity* μ of T is the gcd of the numbers of cycles in each of these three families and the *rotation numbers* of T are those of the cycles. The pair (q,μ) of the three-element set $q\subseteq\mathbb{Q}\cup\{\infty\}$ of rotation numbers and the multiplicity μ of T is called the *geometric code* of T. Similarly to QAs, $T(q,\mu)$ will denote any TA or the set of TAs with geometric code (q,μ).

We arrange the elements of q in an arbitrary order: $q=\{q_1,q_2,q_3\}$, and write the q_i in reduced form: $q_i=m_i/p_i, m_i,\ p_i\in\mathbb{Z}$, $\gcd(m_i,p_i)=1$, $i=1,2,3$. Clearly, $q_i=m_i/p_i=-m_i/(-p_i)$ are two different representations of q_i in reduced form, and this is the only ambiguity in the definition of m_i, p_i. (Of course, $\infty=\frac{1}{0}=-\frac{1}{0}$.) The structure of T depends on the intersection properties of its geodesic cycles. Lemma 1 suggests that we study the vector $\boldsymbol{\lambda}$ defined in this way:

$$\boldsymbol{m}=\langle m_1,m_2,m_3\rangle,\quad \boldsymbol{p}=\langle p_1,p_2,p_3\rangle,$$

$$\boldsymbol{\lambda}=\langle\lambda_1,\lambda_2,\lambda_3\rangle=\boldsymbol{m}\times\boldsymbol{p}=\langle m_2p_3-m_3p_2,m_3p_1-m_1p_3,m_1p_2-m_2p_1\rangle.$$

Then $\lambda_i\neq 0$, $i=1,2,3$, since $q_i\neq q_j$ if $i\neq j$. The definitions of $\boldsymbol{m}$,$\boldsymbol{p}$, and $\boldsymbol{\lambda}$ presume an ordering of the rotation numbers and a choice of the signs of m_i and p_i. Let us fix the order. If we switch, say, from m_I, p_I to $-m_I,-p_I$, then each component of $\boldsymbol{\lambda}$ with index different from I changes its sign. Therefore, by choosing the signs of the m_i and the p_i we can achieve that the components of $\boldsymbol{\lambda}$ are all of the same sign. A simple transposition $m_I\leftrightarrow m_J, p_I\leftrightarrow p_J, I\neq J$ will result in the corresponding transposition $\lambda_I\leftrightarrow\lambda_J$ of the components of $\boldsymbol{\lambda}$, combined with multiplication of all components of $\boldsymbol{\lambda}$ by -1. Consequently, the *set* of rotation numbers determines only the *multiset,*denoted by Λ, of absolute values of the components of $\boldsymbol{\lambda}$.

The following positive integer g, defined in terms of $\boldsymbol{\lambda}$, will play an important role:

$$g=\gcd(\lambda_1,\lambda_2,\lambda_3). \tag{1}$$

Lemma 4 below is fundamental in the proof of Theorem 5, the main result of this section.

Lemma 4. *The number of q_i-cycles in $T(q_1,q_2,q_3,\mu)$ is $\mu_i=\mu|\lambda_i|/g$. There are $\mu|\lambda_1\lambda_2\lambda_3|/|\lambda_i g|$ vertices on each q_i-cycle. The total number of vertices is $n=\mu^2|\lambda_1\lambda_2\lambda_3|/g^2$.*

Proof. By Theorem 3, the total number of vertices is $n=\mu_1\mu_2|\lambda_3|=\mu_2\mu_3|\lambda_1|=\mu_3\mu_1|\lambda_2|$, the number of vertices in each of the three QAs determined by $T(q_1,q_2,q_3,\mu)$. Consequently, $\mu_i=M|\lambda_i|/g$, $i=1,2,3$, for some $M\in\mathbb{N}$ by (1). Then $M=\mu$. The rest follows from this and from the beginning of the proof. $\square$

Theorem 5. *Two tilings $T(q,\mu)$ and $T(q',\mu')$ are shifts of each other if and only if $q=q'$, $\mu=\mu'$. Every pair (q,μ), where q is a three-element subset of $\mathbb{Q}\cup\{\infty\}$ and $\mu\in\mathbb{N}$, is the geometric code of a* TA.

Proof. Just as in the case of QAs, the first statement is quite obvious (but now Lemma 4 is needed). We prove the second statement. Given a three-element subset $q=\{q_1,q_2,q_3\}$ of $\mathbb{Q}\cup\{\infty\}$ with representations $q_i=m_i/p_i, \gcd(m_i,\, p_i)=1,\ i=1,2,3,$ chosen, and given $\mu\in\mathbb{N}$, we consider $Q=Q(q_2,q_3,q_i\to\mu|\lambda_i|/g,\ i=2,3)$. The vector of a q_i-cycle of Q can be lifted to the plane vector $\langle p_i,m_i\rangle$. By Theorem 3, there are $\mu|\lambda_1\lambda_j|/g$ vertices on such a cycle, where $j\neq i,\ i,j=2,3$. Therefore, the side vectors of a parallelogram in Q can be chosen to be

$$\frac{g}{\mu\lambda_1\lambda_2}\langle p_3,m_3\rangle,\qquad \frac{g}{\mu\lambda_1\lambda_3}\langle p_2,m_2\rangle.$$

The diagonal vector that is the sum of these vectors is (in the next-to-the-last equality we use the orthogonality $\boldsymbol{\lambda}\perp\boldsymbol{p},\boldsymbol{m}$)

$$\begin{aligned}\boldsymbol{s}&=\frac{g}{\mu\lambda_1\lambda_2\lambda_3}(\lambda_3\langle p_3,m_3\rangle+\lambda_2\langle p_2,m_2\rangle)\\&=\frac{g}{\mu\lambda_1\lambda_2\lambda_3}\langle\lambda_3p_3+\lambda_2p_2,\lambda_3m_3+\lambda_2m_2\rangle\\&=\frac{g}{\mu\lambda_1\lambda_2\lambda_3}\langle-\lambda_1p_1,-\lambda_1m_1\rangle=-\frac{g}{\mu\lambda_2\lambda_3}\langle p_1,m_1\rangle.\end{aligned}$$

Therefore, the slope of $\boldsymbol{s}$ is $q_1{=}m_1/p_1$. If we consider Q with the diagonals parallel to $\boldsymbol{s}$, it becomes a TA T. The rotation numbers of T are obviously q_1,q_2,q_3, and the number of vertices is $\mu^2|\lambda_1\lambda_2\lambda_3|/g^2$, the same as that of Q. Consequently, the multiplicity of T is μ by Lemma 4. $\square$

The theory of affine isomorphisms of TAs goes parallel with that of QAs. We have $\alpha_P\alpha_UT(q,\mu)=T(Uq,\mu)$, $P\in\mathscr{T}$, and $\bigcup_U T(Uq,\mu)$ is the affine isomorphism class of $T(q,\mu)$.

The definition of the codes of a TA in [20] is equivalent to the following: Let $T=T(q,\mu)$, $q=\{q_1,q_2,q_3\}$, $q_I=m_I/p_I,\gcd(m_I,\,p_I)=1,\ I=1,2,3$. Let i,j,k be a permutation of $1,2,3$. We choose the signs of m_i, p_i and m_j, p_j in such a way that $\lambda_i\lambda_j>0$ ($=$ the components, with respect to $\langle p_i,m_i\rangle,\langle p_j,m_j\rangle$, of the third vector $\pm\langle p_k,m_k\rangle$ have the same sign). Clearly, such $\langle p_i,m_i\rangle,\langle p_j,m_j\rangle$ exist, and then $\langle -p_i,-m_i\rangle,\langle -p_j,-m_j\rangle$ can also be chosen, and these are the only choices. The *affine* code (l_i,c_{ij},μ_i) of T is then defined like for QAs, using the orientations of $\mathfrak{C}_i,\mathfrak{C}_j$ determined by the vectors $\langle p_i,m_i\rangle,\langle p_j,m_j\rangle$ (or by their negatives). Now c_{ij} is uniquely defined. By Lemma 4, $l_i=\mu|\lambda_j\lambda_k|/g$, $\mu_i=\mu|\lambda_i|/g$, and thus $c_{ij}=\mu_jr_{ij}(=l_ir_{ij}/|\lambda_k|)$, where r_{ij} is determined by Lemma 2 like for a QA.

5. Affine isomorphism of TAs when $g = 1$

If U is a 2×2 unimodular integer matrix, then for any TAs $T(m_1/p_1, m_2/p_2, m_3/p_3, \mu)$, $T(m_1'/p_1', m_2'/p_2', m_3'/p_3', \mu')$ with $\langle m_i', p_i'\rangle^t = U\langle m_i, p_i\rangle^t$ we obtain by easy calculation that $\boldsymbol{\lambda}' = (\det U)\boldsymbol{\lambda} = \pm\boldsymbol{\lambda}$. Consequently, *if two TAs are affine isomorphic, then they have the same multiset Λ*. We show that the converse is also true for TAs with a given μ if $g = 1$. Motivated by (3) below, we also show that three pairwise relatively prime positive integers can always be represented as the multiset Λ for some TA. These results are stated in the forthcoming Proposition 7, which follows immediately from the next lemma. At the end of the section we consider the case $g \neq 1$.

Lemma 6. (i) *Let $\mathscr{C}$ be an affine-isomorphism class of* TA*s and let $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$ be an ordering of the multiset Λ of $\mathscr{C}$. Then $\mathscr{C}$ contains a $T(\infty, q_2, q_3, \mu)$, $0 \leqslant q_2 < 1$, such that q_i, $i = 2, 3$, can be written as $q_i = m_i/p_i$*, $\gcd(m_i, p_i) = 1$, *with $\langle 1, m_2, m_3\rangle \times \langle 0, p_2, p_3\rangle = \boldsymbol{\Lambda}$ or equivalently, with*

$$p_2 = \Lambda_3, \quad p_3 = -\Lambda_2, m_2\Lambda_2 + m_3\Lambda_3 = -\Lambda_1. \tag{2}$$

The signs of $m_i, p_i, i = 1, 2$, are determined uniquely by the q_i. If the components of $\boldsymbol{\Lambda}$ are distinct, then $T(\infty, q_2, q_3, \mu)$ is determined uniquely by its properties.

(ii) *In any $T(q_1, q_2, q_3, \mu)$,*

$$g = \gcd(\Lambda_1, \Lambda_2) = \gcd(\Lambda_2, \Lambda_3) = \gcd(\Lambda_3, \Lambda_1). \tag{3}$$

(iii) *Every vector $\boldsymbol{\Lambda} = \langle \Lambda_1, \Lambda_2, \Lambda_3\rangle$ of pairwise relatively prime nonzero integers can be written in a unique manner as $\boldsymbol{\Lambda} = \boldsymbol{m} \times \boldsymbol{p}$, where* $\gcd(m_i, p_i) = 1$, $p_1 = 0$, $m_1 = 1$, $0 \leqslant m_2/p_2 < 1$.

Proof. (i) Let $T = T(\hat{m}_1/\hat{p}_1, \hat{m}_2/\hat{p}_2, \hat{m}_3/\hat{p}_3, \mu) \in \mathscr{C}$, and let the $\boldsymbol{\lambda}$ of T equal the given $\boldsymbol{\Lambda}$. By the beginning of this section, $N^+M^{-1}T$ possesses the properties required of $T(\infty, q_2, q_3, \mu)$ if M is the matrix of the column vectors $\langle \hat{m}_1, \hat{p}_1\rangle^t$, $\langle \hat{m}_2, \hat{p}_2\rangle^t$ and N^+ is determined for this M by Lemma 2. The signs of $m_i, p_i, i = 1, 2$, are uniquely determined by (2) because m_1, p_1, and $\boldsymbol{\lambda}$ are given for $T(\infty, q_2, q_3, \mu)$. If the components of $\boldsymbol{\Lambda}$ are distinct, then their order determines an order of the families of geodesic cycles of each element of $\mathscr{C}$. Hence, the uniqueness of $T(\infty, q_2, q_3, \mu)$ follows from the beginning of this section combined with Lemma 2.

(ii) This can be proved directly but it also follows easily by symmetry from the consequence $\gcd(\Lambda_2, \Lambda_3) | \Lambda_1$ of (2).

(iii) Since $\gcd(\Lambda_2, \Lambda_3) = 1$, (2) has integer solutions. For every solution we have $\gcd(m_2, \Lambda_3) = \gcd(m_3, \Lambda_2) = 1$ because $\gcd(\Lambda_2, \Lambda_1) = \gcd(\Lambda_3, \Lambda_1) = 1$. Consequently, $\gcd(m_2, p_2) = \gcd(m_3, p_3) = 1$ for every solution. Therefore, it is sufficient to show that there is a unique solution of (2), such that $0 \leqslant m_2/p_2 < 1$. If $(\overline{m_2}, \overline{m_3})$ is a solution, then all solutions (m_2, m_3) are given by $m_2 = \overline{m_2} + k\Lambda_3$, $m_3 = \overline{m_3} - k\Lambda_2$, $k \in \mathbb{Z}$. The assertion follows easily from this, since $p_2 = \Lambda_3$. $\square$

Proposition 7. *For every multiset $\Lambda=\{\Lambda_1,\Lambda_2,\Lambda_3\}$ of pairwise relatively prime positive integers and for every $\mu\in\mathbb{N}$ there exists a* TA *whose multiset $\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}$ is Λ and whose multiplicity is μ. All such* TA*s for given Λ,μ are affine-isomorphic to each other.*

Corollary 8. *Two* TA*s are affine isomorphic if they have the same set of numbers of the three kinds of cycles, provided that in both* TA*s the* gcd *of the numbers of the three kinds of cycles is equal to the* gcd *of the numbers of vertices on single cycles.*

Proof. Corollary 8 is an easy consequence of Proposition 7 because Lemma 4 implies that in every TA, $\gcd(l_i,l_j)=\mu\Lambda_k$ for any permutation i,j,k of $1,2,3$, and thus $\gcd(l_1,l_2,l_3)=\mu g$. $\square$

In the general case $\gcd(\Lambda_i,\Lambda_j)=g$, $i\neq j$, solutions of (2) exist and every solution m_2,m_3 can be obtained from one solution $\overline{m_2},\overline{m_3}$ by the formula

$$m_2=\overline{m_2}+\frac{\Lambda_3}{g}k,\quad m_3=\overline{m_3}-\frac{\Lambda_2}{g}k,\quad k\in\mathbb{Z}.$$

Consequently, there are g solutions of (2) with $0\leqslant m_2/p_2<1$. However, these provide TAs only if $\gcd(m_2,\Lambda_3)=\gcd(m_3,\Lambda_2)=1$. By Lemma 6(i) (existence), the number of affine-isomorphism classes of TAs with given μ and Λ is *at most g*.

If $\Lambda_1=2,\Lambda_2=10,\Lambda_3=6$, then (2) implies $5m_2+3m_3=-1$, all solutions of which are $m_2=1+3k$, $m_3=-2-5k$, $k\in\mathbb{Z}$. Therefore, either m_2 or m_3 is even, and thus no TA exists with this $\boldsymbol{\Lambda}$ by Lemma 6(i) (existence). However, for

$$\begin{aligned}&p_1=0,\quad p_2=15,\quad p_3=-9,\quad m_1=1,\quad m_2=13,\quad m_3=-8,\\&p_1'=0,\quad p_2'=15,\quad p_3'=-9,\quad m_1'=1,\quad m_2'=8,\quad m_3'=-5,\end{aligned}$$

we obtain

$$\boldsymbol{\Lambda}=\boldsymbol{m}\times\boldsymbol{p}=\boldsymbol{\Lambda}'=\boldsymbol{m}'\times\boldsymbol{p}'=\langle 3,9,15\rangle.$$

Therefore, for $\boldsymbol{\Lambda}=\langle 3,9,15\rangle$ there is more than one solution of (2) with $0\leqslant m_2/p_2<1$ that provides TAs with a given μ. These TAs cannot be affine isomorphic by Lemma 6(i) (uniqueness).

6. The theory of HAs by duality

The notion of a HA can also be defined intrinsically: A HA is a tiling of $\mathscr{T}$ with hexagons: each hexagon is assembled from congruent copies of a triangle as to form a TA, and then the hexagonal tiles are adjoined likewise. Equivalently, a HA is a tiling of $\mathscr{T}$ with congruent hexagons in parallel position; each of the three diameters of a hexagon is parallel to and twice as long as the sides whose endpoints are different from those of the diameter. We do not allow all hexagons with opposite sides parallel and of equal length because then the second dual of a HA may become different from the original tiling and the theorem below becomes untrue.

The theory of HAs follows immediately from that of TAs by taking duals and then duals of duals. The parameters (q_1, q_2, q_3, μ) and the codes (l_i, c_{ij}, μ_i) of a HA are defined to be those of the dual tiling, and we use the notation $H(q_1, q_2, q_3, \mu)$ similarly to that for QAs and for TAs.

Theorem 9. *Theorem* 5 *remains true if* TA *is replaced with* HA *and* T *with* H.

The theory of affine isomorphisms of TAs also extends to HAs because the second dual is the original tiling.

7. Regular toroidal maps

A regular toroidal map is defined to be the homeomorphic image of a regular affine (or equivalently, of a regular euclidean) tiling of a flat torus. There is also an intrinsic definition: A *map* on a closed surface is a finite *graph* imbedded on the surface, such that each connected component (open face) of the surface minus the graph is homeomorphic to an open disc. Each vertex of the graph may be connected with any (not necessarily distinct) vertex by any number of edges. The *valence* of a vertex is the number of edge ends containing the vertex. A *dual* map has a vertex inside each open face of the original map, and these vertices are connected, without crossings, by simple curves across common edges of the faces. A map is said to be of *type* (p,q) if every vertex has valence q and every vertex of the dual map has valence p. Obviously, any dual of a map of type (p,q) is of type (q,p). The closed faces of a map of type (p,q) are (topological) p-gons if edges incident to only one face are counted twice. On a torus only the types $(6,3)$, $(3,6)$, and $(4,4)$ exist [2, p. 201]. A map admitting a type is said to be *regular* or sometimes *quasi-regular* or *Platonic*.

A *map-isomorphism* between two maps is a homeomorphism between two surfaces that, together with its inverse, sends vertices to vertices, edges to edges, faces to faces, and preserves incidence. The existence of a map-isomorphism between two maps is equivalent to the existence of a *combinatorial equivalence* between the underlying two-dimensional convex cell complexes [6].

Clearly, every regular affine tiling of a flat torus is a regular toroidal map. Conversely, every regular toroidal map defined by the intrinsic definition, is map-isomorphic to a regular affine tiling of a flat torus. This result was proved by Altshuler [1], using the following fundamental observation: A toroidal map of type $(3,6)$ (of type $(4,4)$) decomposes into nonintersecting *normal cycles* in 3 (2) ways. A *cycle* is a nonoriented simple closed path and a *normal cycle* is a cycle that leaves, at each of its vertices, exactly one (two) edges on the right in the case of type $(4,4)$ (type $(3,6)$). We extend this definition to the type $(6,3)$ by saying that a cycle in such a map is *normal* if it leaves, at each of its vertices, one edge on alternating sides of the cycle. Every regular toroidal map of type (6, 3) decomposes into nonintersecting normal cycles in three ways.

In this section we study the *isotopy* of regular toroidal maps. For this we must consider all maps on *one and the same torus* $\mathscr{T}$. Two maps on $\mathscr{T}$ are said to be *isotopic* to each other if they are map-isomorphic through a self-homeomorphism of $\mathscr{T}$ that is isotopic to the identity homeomorphism (see the Preliminaries). Each map-isomorphism class splits into infinitely many *isotopy classes*. Furthermore, it is easy to see that duals of isotopic (map-isomorphic) maps are isotopic (map-isomorphic). It follows that any second dual is isotopic to the original map.

For the types (3, 6) and (6, 3) we define the *multiplicity* of a map to be the greatest common divisor of the numbers of cycles in each of the three families of nonintersecting normal cycles. In the case of type (4, 4) the *multiplicity* assigns, to each of the two families of nonintersecting normal cycles, the number of cycles in the respective family.

The first statement of the theorem below follows immediately from the results in the last paragraph of the Preliminaries, which results enable us to turn the theory of regular affine tilings of a flat torus into a theory of regular maps on a topological torus. This theory is summarized as follows.

Theorem 10. *Every regular map M on $\mathscr{T}$ is isotopic to a regular affine tiling T_M of $\mathscr{T}$. The tiling T_M is determined up to shifts of $\mathscr{T}$. Two regular $\mathscr{T}$-maps of the same type are isotopic if and only if the normal cycles of one map are isotopic to those of the other, and the multiplicities of the maps are the same.* (*In the* (4, 4)-*case, the latter means that isotopic normal cycles in the two maps have the same multiplicity.*) *Two regular $\mathscr{T}$-maps of the same type are map-isomorphic if and only if* (1) *there is a self-homeomorphism of $\mathscr{T}$ that maps the isotopy classes of the normal cycles of one map to those of the other* (2) *the multiplicities of the maps are the same.* (*In the* (4, 4)-*case, the latter means that the normal cycles of the two maps, connected by the homeomorphism, have the same muliplicity.*)

Corollary 11. *Let a "topological coordinate system" Ξ_t, Υ_t be specified on $\mathscr{T}$. Then the existence of a self-homeomorphism of $\mathscr{T}$ that maps the isotopy classes of the normal cycles of one map to those of another map is equivalent to the existence of a 2×2 unimodular integer matrix U that transforms the rotation numbers of the normal cycles of one map to those of the other.*

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